Math 4650
Topic 2a - Application to calculating square roots

Application to finding square roots

Theorem: Let $a > 0$ be a real number.

Define the sequence:

$a_1 =$ any positive real number

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \text{ for } n \geq 1$$

Note: Formula comes from Newton's method

Then:

① $(a_n)_{n=1}^{\infty}$ converges

② $\lim_{n\to\infty} a_n = \sqrt{a}$

③ $|a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n}$ when $n \geq 2$ ← error bound

Proof:

① We need some facts first.

Fact (i) $a_n > 0$ for $n \geq 1$:

We are given $a_1 > 0$ and $a > 0$

Assume $a_k > 0$.

Then, $a_{k+1} = \frac{1}{2}\left(\underset{\substack{\uparrow \\ a_k > 0}}{a_k} + \underset{\substack{\uparrow \\ a/a_k > 0}}{\frac{a}{a_k}}\right) > 0$

By induction, $a_n > 0$ for all n.

Fact (ii): $a_n \geq \sqrt{a}$ for $n \geq 2$

Let $n \geq 1$.

By def we have $2a_{n+1} = a_n + \frac{a}{a_n}$.

So, $a_n^2 - 2a_n a_{n+1} + a = 0$.

Thus, $x^2 - 2a_{n+1}x + a = 0$ has a real root $(x = a_n)$.

So, the discriminant must be non-negative.

That is, $4a_{n+1}^2 - 4a \geq 0$.

So, $a_{n+1}^2 \geq a$.

Thus, $a_{n+1} \geq \sqrt{a}$ for $n \geq 1$.

We used fact (i) here also.

Fact (iii): $a_n \geq a_{n+1}$ for $n \geq 2$

Let $n \geq 2$

Then,

$$a_n - a_{n+1} = a_n - \frac{1}{2}a_n - \frac{1}{2}\frac{a}{a_n} = \frac{1}{2}\left(\frac{a_n^2 - a}{a_n}\right) \geq 0$$

Fact (ii): $a_n^2 \geq a$
So, $a_n^2 - a \geq 0$
Fact (i): $a_n > 0$

Thus, $a_n \geq a_{n+1}$

Now we use the above to prove ①

We have shown that

$$a_2 \geq a_3 \geq a_4 \geq a_5 \geq \ldots\ldots\ldots \geq \sqrt{a} > 0$$

By the monotone convergence theorem,

$(a_n)_{n=1}^{\infty}$ converges.

(2) Let $L = \lim_{n\to\infty} a_n$.

We know $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ for $n \geq 1$.

Taking the limit of both sides gives

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

So, $2L^2 = L^2 + a$

Thus, $L^2 = a$.

And since $L \geq 0$

we must have $L = \sqrt{a}$.

This is from HW 2.
If $\lim_{n\to\infty} x_n = L$ where $x_n \geq 0$ for all n, then $L \geq 0$

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(3) Let $n \geq 2$.

Then, $a_n \geq \sqrt{a} \geq \frac{a}{a_n}$

fact (ii) above

$a_n \geq \sqrt{a} = \frac{a}{\sqrt{a}}$

So, $a_n \geq \frac{a}{\sqrt{a}}$

Thus, $\sqrt{a} \geq a/a_n$

Thus,

$$0 \leq a_n - \sqrt{a} \leq a_n - \frac{a}{a_n} = \frac{a_n^2 - a}{a_n}$$

So, $|a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n}$

$\blacksquare$

Ex: Let's approximate $\sqrt{2}$. Here $a=2$.

Set $a_1 = 1 > 0$.

We have:

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$	Error bound $\frac{a_n^2 - a}{a_n}$
$a_2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{2} = 1.5$	$\frac{a_2^2 - 2}{a_2} = \frac{1.5^2 - 2}{1.5} \approx 0.1666\ldots$
$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{(3/2)}\right) = \frac{17}{12}$ $\approx 1.416666\ldots$	$\frac{a_3^2 - 2}{a_3} = \frac{1}{204} \approx 0.00490196\ldots$
$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408}$ $\approx 1.414215686\ldots$	$\frac{a_4^2 - 2}{a_4} = \frac{1}{235{,}416}$ $\approx 0.000004247796\ldots$
$a_5 = \frac{1}{2}\left(\frac{577}{408} + \frac{2}{577/408}\right)$ $= \frac{665857}{470832} \approx 1.41421356237\ldots$	$\frac{a_5^2 - 2}{a_5} = \frac{1}{313{,}506{,}783{,}024}$ $\approx 3.1897 \times 10^{-12}$

We get rapid convergence here.